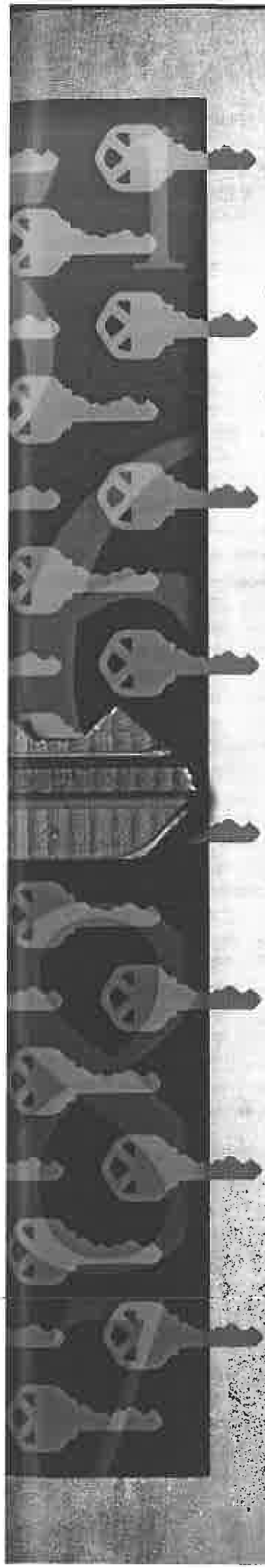

Problems before Procedures: Systems of Equations

Students today come to first-year algebra with considerable prior experience and a wide range of skills. Teachers need to modify their instructional strategies accordingly.

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As a new teacher in the 1980s, I savored my work with ninth graders who were studying beginning algebra. I especially enjoyed the chance to excite young minds about the wonder of variables and the power of generalization. We programmed graphing calculators (a tool most of them were encountering for the first time) and explored the role of mathematics in other disciplines, especially art and science. My ninth graders were proud to study algebra. They approached the subject with some trepidation but also with curiosity. What was this algebra thing they had heard so much about?

That was then, this is now. Today, beginning algebra in the high school setting is associated more with remediation than pride. Students enroll by mandate and attend under duress. Class rosters in this "graveyard" course, as it is often referred to, include sophomores and juniors who are attempting the course for the second or third time. Even the ninth graders have seen many of the ideas in an earlier context because of the increasing prevalence of seventh- and eighth-grade mathematics courses designed to frontload first-year algebra topics. As a result, the content can feel like review, even when it is not. Such circumstances present unique challenges for teachers and students alike.

My experience as a mathematics educator leads me to a single principle that can consistently lead to successful teaching and learning in an early algebra class: Pose problems before presenting procedures. This is not a new idea. However, it is one that many teachers either

Table 1 Record of Student Guesses

First number	20	21	22	23	24	25	26
Second number	15	14	13	12	11	10	9
Sum	35	35	35	35	35	35	35
Difference	5	7	9	11	13	15	17

forget or opt not to implement under the pressures of time and topic coverage. Especially when accompanied by instructional practices that prioritize student communication and collaboration, this subtle shift in practice can prove particularly valuable in the beginning algebra setting. Compelling problems provide a powerful tool for assessing prior knowledge, encourage students to generate mathematical ideas, and pave the path to symbolic representations. Using problems to drive mathematics instruction also levels the playing field by valuing powerful thinking over memorization, thus fostering a community of learners in which everyone has something to contribute. This article uses systems of linear equations to illustrate how this strategy might unfold in a classroom.

ASSESSING PRIOR KNOWLEDGE: THE MYSTERY NUMBERS PROBLEM

Students bring extensive prior knowledge to beginning algebra, much more than even they know that they possess. The challenge is to uncover this understanding in ways that students find safe, empowering, and relevant to the topic at hand. In the case of systems of equations, students will likely have had some recent experience with equations in two variables. They should also have a sense of the multiple ways to represent such relationships: tables, graphs, symbols, and verbal description.

Many textbooks begin immediately with symbols. Instead, we start with a problem like the Mystery Numbers problem:

> I'm thinking of two numbers. Their sum is 35, and their difference is 13. What are the numbers?

As teachers, we recognize that the Mystery Numbers problem lends itself to representation with two equations and two variables. However, whether or not students have prior experience in solving systems of linear equations, they will take a nonsymbolic approach to this problem.

In class, I recommend beginning with a think-pair-share strategy: Give each student some time to consider the problem alone and to jot down initial ideas. Then organize students in pairs to share their thinking and work toward a solution. Ask pairs to keep track of the strategies they discuss, focusing on process at this point. For pairs that seem to finish quickly, ask how many solution methods they can find.

Once all pairs of students seem to have found a solution, move to pair consultations: Have each pair meet briefly with another to share strategies and solutions. These small groups should select one reporter who can communicate the collective ideas during the whole class discussion that will follow. As groups report out, teachers can provide a public posting place or "landing pad"—such as poster paper, whiteboard, or SMART Board®, which can be displayed and added to during the conversation as well as saved for future reference.

For the Mystery Numbers problem, students will likely describe a process of comparing number pairs, such as 20 and 15, that add up to 35 and then finding the difference, making adjustments until they find the right combination. With some encouragement, either during pairs work or as part of the whole-class debriefing, students might record their thinking in a table. If they do not, then the teacher can suggest making a table to keep track of the number pairs that students have tested (see **table 1**). The table will encourage new questions: What patterns do students see in the table? What do they notice about the relationship between the first number and the second number? How does one affect the other? Does this problem have many answers or just one? How do you know?

The purpose of carefully examining the table is to ensure that students get some sense of how all the possible mystery numbers are related to one another and why only one pair can serve as the answer to the given problem. Students can use this thinking to represent the situation symbolically.

If the class discussion indicates that students have used a systematic guess-and-check strategy exclusively, then teachers can ask students how they might use variables to solve this problem. Pairs can then work together to represent each of the given facts with an equation using variables—one for each fact. In situations such as this, index cards work well as a reporting tool. On the index card, pairs write down the equations they come up with, along with their names. Teachers can collect these index cards as they move about the room, make a quick formative assessment, and decide where they want to take the group next. (Index cards can also be used at the end of the class period as an exit ticket to provide assessment data that will inform the design of the next day's lesson.)

If, on the other hand, a significant number of students immediately solve the problem using two equations and an algebraic strategy (such as elimination or substitution), ask students to explain how their strategy works. Remind them how important it is to understand every procedure or algorithm they choose to use. Try to distinguish between what students know and what they have simply memorized.

In either case, students may benefit from exploring one or two more similar problems before making the move to symbols. Here are two possibilities:

> I'm thinking of two numbers. Their sum is 111, and their difference is 55. What are the numbers?

> I'm thinking of two numbers. The larger number is double the smaller one. Their sum is 96. What are the numbers?

GENERATING MATHEMATICAL IDEAS

The ultimate goal of posing problems before presenting procedures is to help students generate mathematical ideas based on their own thinking rather than telling them what to think. With the Mystery Numbers problem, students need to see for themselves that two conditions must be satisfied simultaneously: The numbers must add up to 35 and have a difference of 13. Infinitely many pairs satisfy each condition individually, but only one pair satisfies both conditions at the same time. If students have prior experience working with variables and writing equations, then (perhaps with some gentle nudging and careful questioning) they should be able to represent the two conditions with two equations:

$$x + y = 35$$
$$x - y = 13$$

Once the class has generated the equations above and agreed on their meaning, including what the variables represent, students can begin using their mathematical know-how to determine the values for x and y that make these equations true at the same time. In their pairs, students can talk about this idea for a few minutes. What possibilities do they see? Before pairs report out, divide the class in half. Let students know that half the pairs will pursue a graphing strategy, while the other half will take a more algebraic approach. As a class, brainstorm helpful hints for each group. (The activity described here is intended for a long block of eighty to ninety minutes.)

Some ideas that teachers will want to help elicit for a graphing approach include these:

- Both equations can be graphed as lines.
- Consider rewriting the equations in a different form to help with graphing.
- What do you see when both lines are graphed on the same set of axes?

Some ideas that teachers will want to help elicit for an algebraic approach include these:

- Rewrite the equations to see whether you can connect them in some way.
- Use properties of equality. Example: If $a = b$ and $c = d$, then $a + c = b + d$.
- If you find the value of one variable, how can it help you find the other?

Students can test and record their strategies while the teacher circulates to ask clarifying questions and provide support (see **fig. 1**).

Student pairs likely will generate powerful ideas here that clearly point to the procedures traditionally taught for solving a system of two linear equations. Pairs can write up their work on an 11 × 17 piece of paper, creating a miniposter that shows the connection between their numeric solution to the problem and their algebraic or graphing solution as well as any questions that they have. If time permits, pairs can consult with one another and hang their miniposters on one wall of the classroom for a gallery walk, in which students walk past each poster, giving the class a greater sense of its

Graphing Approach	Algebraic Approach
• How do you know that the graphs and the equations represent the same set of numbers? • Why does it make sense that your lines intersect? • How can you anticipate an intersection by looking at the equations? • Where do your lines intersect? • What is the significance of this point?	• Can you remind me what x and y represent in these equations? • Do x and y have the same meaning in both equations? How do you know? • How might you combine the two equations to make one equation? • Explain in words what this statement means: "If $a = b$ and $c = d$, then $a + c = b + d$." • How does this statement apply to our equations?

Fig. 1 Teachers can ask questions such as these to elicit understanding.

collective work. A short whole-class discussion wraps up the lesson.

BUILDING TOWARD UNDERSTANDING OF PROCEDURES

As the work continues in the days ahead, a focus on solving incrementally more difficult problems will verify and enrich student conjectures. Delaying the presentation of procedures ultimately creates a situation in which articulating algorithms brings together the collection of student-initiated ideas that have surfaced in class discussion. When students feel empowered to generate the mathematics themselves, rather than simply memorize or repeat what others have created, mathematics takes on new meaning, and students become intrinsically motivated to learn (Middleton and Jansen 2011). In addition, the problem-solving process enables students to consider techniques that rarely surface in a class focused first on procedures and then on problems as applications of those procedures.

The following example is based on the Farmer's Market problem, which appeared in *Algebra and Algebraic Thinking in School Mathematics* (Ferrucci et al. 2008).

> At the local Farmer's Market, you bought 7 apples and 4 peaches for $4.80. Your friends bought 5 apples and 2 peaches for $3.00. When you got home, your sister asked how much each fruit cost. You don't remember, but you can use mathematics to figure it out. (Adapted from Ferrucci et al. 2008, p. 196)

This problem and others like it demonstrate how elementary school students can use the Singapore model method to solve a system of equations problem without symbols (Ferrucci et al. 2008). When older students are given this problem before formal

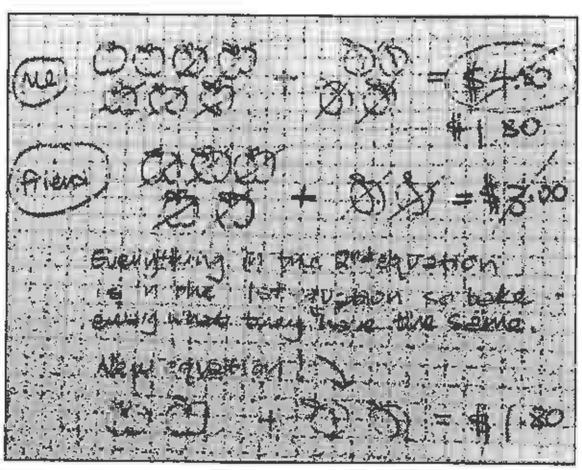

Fig. 2 A student-generated method demonstrates powerful preprocedure thinking.

instruction about solving systems of linear equations, they generate sketches and other nonsymbolic approaches akin to the model method (see **fig. 2**).

This student's ideas taught me a profound lesson about the importance of making space for mathematical creativity through problem solving. With some encouragement and probing questions, ~~she ultimately solved the~~ problem using her "everything in this equation is in the other equation" approach. "This is legal, right?" she asked with excitement. Her conclusion: Three apples cost $1.20, so one apple cost 40 cents, and (after a short pause) a peach costs 50 cents. As this student reflected on her work, she also noted that her "new" equation could have been simplified:

cost of apple + cost of peach = 90 cents

As I strive to make more room in my teaching for this kind of student thinking to occur, I marvel at the new insights I gain from my students. High-achieving students take note of their peers' methods, many of which involve highly efficient mathematical reasoning before symbolic manipulation.

DRIVING THE MOVE TO SYMBOLS

Ongoing opportunities to solve increasingly complex problems—for which strategies such as drawing diagrams, making tables, and engaging in an informed guess-and-check strategy become less practical—will stimulate student interest in more systematic and globally applicable techniques. In addition, ongoing experience as cooperative problem solvers, who readily incorporate the thinking of others into their own ideas, will enable students to view the procedures as culminating strategies, tools that encompass the thinking of many people rather than a single person—namely, their teacher.

The Spare Change Problem

This is a highly adaptable problem that resonates with students in a range of settings. It is concrete and tangible, with multiple entry points. The larger the number of coins, the more students will employ strategies using variables.

> Asa saves dimes and quarters so that he will have exact change for the bus. He dumps out the container that he has been using to collecting his coins and finds that he has 47 coins that are worth $9.50. How many of each coin does he have?

I especially like the way students see the connection between the two variables in this problem. Even those generally trying to avoid symbols create detailed tables to demonstrate how they can exchange 2 quarters for 5 dimes.

The Theater Ticket Mystery Problem

This is another problem that nudges students toward representing ideas symbolically and articulating some general principles for combining two equations in two variables.

> Last night was a sellout for the school play. However, the students running the box office lost track of the number of adult tickets and children's tickets that were sold. Here's what you know: The theater seats 200 people. Adult tickets were priced at $8, and children's tickets were $5. At the end of the night, the box office had collected $1360 in ticket sales. How many children and how many adults saw the show last night?

As students begin working more consistently with symbols, one common challenge with both the Spare Change and the Theater Ticket Mystery problems is that some students resist representing them with two equations. It is as if they simply do not see the problem in this way. Instead of writing a separate equation for the total number of coins or tickets, they make a mental substitution, generating a single equation in one variable. For example, they might write $25q + 10(47 - q) = 950$ for the Spare Change problem and $8A + 5(200 - A) = 1360$ for the Theater Ticket Mystery problem.

Such students do not seem to see the total number of coins or the total number of seats as facts that warrant a separate equation. It is important for teachers to understand and honor this preference. However, it is equally critical to help students recognize the value of using two equations.

In my own teaching, for each problem that students explore, we try to generate at least two, preferably three, solution methods—the more, the better. Doing so helps students see how, even in algebra, there is more than one way to arrive at the correct answer. It also creates an opportunity to discuss how the "one-equation" strategy could be viewed as a "two-equation" strategy, with some elements completed mentally rather than on paper. As the exploration of linear equations continues, well-crafted distance problems—preferably based on student experiences (such as traveling upstream versus downstream) or data collected in class (such as using stopwatches to compare student rates of walking, jumping, and running)—also help students readily embrace the two equations in two-variables strategy.

REAPING THE BENEFITS

Letting problems serve as the leading edge for mathematics instruction takes courage and careful planning, especially in a first-year algebra class. Many teachers fear that they will not be able to cover the same amount of material; they believe that presenting procedures first ensures that all students "see the material." In my experience, students do not learn the topics that we "cover"; they are much more likely to remember and retain the mathematics that they do and create themselves. Doing mathematics means solving problems—that is, solving compelling mathematical questions to which one does not immediately know the answer. The exercises that fill most textbooks are not *problems*. As the term *exercises* suggests, these questions rarely involve new thinking on the part of students; they merely serve as practice for procedural knowledge (Johnson and Herr 2001).

Students need something more. They need to think deeply, question decidedly, reason carefully, and verify confidently so that they can come to believe that "learning mathematics involves learning ways of thinking. It involves learning powerful mathematical ideas rather than a collection of disconnected procedures" (Carpenter et al. 2003, p. 1). Rather than viewing mathematics as something done to them, students need to view mathematics as something they can and want to do. Good problems followed by rich classroom interactions can make all the difference.

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